Boundary formulation for three-dimensional anisotropic crack problems

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Boundary integro-differential equations for three-dimensional anisotropic cracked bodies are derived. Both the cases of the infinite body (with an embedded crack) and a finite body with an embedded or surface crack are considered. Detailed mathematical conditions for the results to be valid are specified throughout.

Keywords: anisotropic elasticity, crack, integral equation

1. Introduction

In the field of numerical techniques, the most widely known is undoubtedly the domain finite element method (FEM). In recent years, the structural components engineers have to cope with become more and more complex, and the number of degrees of freedom entailed by a sufficiently refined mesh is continually increasing to obtain more and more accurate results. It was precisely for this reason that the advent of the boundary integral equation method (BIEM) is of great significance since it allows the structure to be meshed over the boundary only, at least when the body forces are absent. In any case, the BIEM allows the solving of problems with smaller matrices because the unknowns are exclusively boundary quantities.

The pioneering works in the BIEM were those of Rizzo,[1] Cruse and Rizzo,[2] and Cruse.[3,4] The regularized expression of the BIE was first given by Rizzo and Shippy.[5] Since then the boundary method has been extensively developed showing its ability to deal with various types of mechanical problems. In the field of fracture mechanics, the BIE, or more precisely the boundary integro-differential equation (BIDE), has been formulated for elastostatic, thermoelastic as well as elastodynamic problems.[6-10] Regularized BIDEs have also been given elsewhere.[11-14] A complete review of different works in the BIE field can be found in the paper of Tanaka et al.,[15] though related to regularization techniques.

As a matter of fact, whereas most of the materials are more or less anisotropic, the BIEM was mainly developed for isotropic materials. This can be accounted for by two reasons. First, equations for the isotropic case are simpler to solve from the theoretical and numerical standpoints alike. Second, the elastic material properties are much more difficult to be determined experimentally in anisotropy. Nevertheless, increasing use is being made of anisotropic structural components, and this requires more efficient studies for this class of materials. Using the decomposition of the Dirac function into plane waves, Vogel and Rizzo[16] derived the integral representation of the fundamental solution for a general anisotropic elastic three-dimensional (3-D) continuum. Later on, an efficient numerical implementation for anisotropic problems was proposed by Wilson and Cruse.[17] In fracture mechanics, Sladek and Sladek[18] and Balas et al.[19] (pp. 50–52) have discussed the boundary formulation for anisotropic cracked bodies, and the corresponding BIDE has been proposed, conjecturing that some results in isotropy can be extended to anisotropy.

In the first part of this paper, the integral representation of the fundamental solution for an anisotropic elastic medium derived by Vogel and Rizzo[16] is briefly reviewed and its basic properties are investigated. In particular, relations describing the limit behavior of the fundamental solution near a closed or open surface are presented. All the results obtained are generalizations of the well-known ones in the isotropic case. As an application, in the second part of the paper, the BIE and BIDE are derived for the problem of anisotropic cracked bodies. The boundary formulation includes both the cases of the infinite body (with an embedded crack) and a finite body with an embedded or surface crack. Throughout the paper, emphasis is made on the mathematical conditions for the results to be obtained.

2. The fundamental solution and its basic properties

In this section, the fundamental solution for a homogeneous anisotropic linear elastic medium is recalled and its basic properties are given. The components of the fourth-order elastic tensor C written in a fixed base $(e_1 e_2 e_3)$ of the 3-D space $\mathscr{E}$ are material constants verifying the usual symmetries:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \tag{1}$$

Let us introduce the following definition where the summation convention is implied over repeated subscripts, which all have the range $(1,2,3)$.

Definition

Given a unit vector $\mathbf{e}_m$ of the base $(e_1 e_2 e_3)$, the fundamental solution related to $\mathbf{e}_m$ and denoted by $U(\mathbf{e}_m, x, y)$ is the solution of the partial differential equation in the infinite space ε:

$$\operatorname{div}[C\colon \operatorname{grad} U(\mathbf{e}_m, x, y)] + \delta(y-x)\mathbf{e}_m = 0 \tag{2}$$

where $\delta(y-x)$ is the Dirac function.

The corresponding stress tensor is defined as:

$$\begin{aligned}\Sigma(e_m, x, y) &= t_{U(e_m, x, y)}(y, n_y = e_j) \otimes e_j \\ &= C\colon \operatorname{grad} U(e_m, x, y)\end{aligned} \tag{3}$$

where the tensor product $a \otimes b$ is defined by $(a \otimes b)_{ij} = a_i b_j$. The differentiation in equation (9) is performed with respect to variable y, and $\mathbf{t}_{U(e_m, x, y)}(y, n_y)$ denotes the stress vector at point y with respect to normal $\mathbf{n}_y$ and corresponding to the displacement field $U(e_m, x, y)$.

Now the fundamental displacement tensor is defined by:

$$\begin{aligned}&E(x,y) = U(e_m, x, y) \otimes e_m, \quad \text{i.e.,} \\ &E_{ij}(x,y) = U_i(e_j, x, y)\end{aligned} \tag{4}$$

This, in turn, gives rise to the third-order tensor of the fundamental stress:

$$\begin{aligned}&D(x,y) = \Sigma(e_m, x, y) \otimes e_m, \quad \text{i.e.,} \\ &D_{ijk}(x,y) = \Sigma_{ij}(e_k, x, y) = C_{ijpq}\frac{\partial E_{pk}}{\partial y_q}(x,y)\end{aligned} \tag{5}$$

Eventually, the Kupradze tensor is defined by[20] (p. 99):

$$\begin{aligned}&T(x, y, n_y) = t_{U(e_m, x, y)}(y, n_y) \otimes e_m, \quad \text{i.e.,} \\ &T_{ik}(x, y, n_y) = C_{ijpq} n_j(y) \frac{\partial E_{pk}}{\partial y_q}(x,y)\end{aligned} \tag{6}$$

The relationships between the above-defined tensors are straightforward:

$$T_{ik}(x, y, n_y) = \Sigma_{ij}(e_k, x, y) n_j(y) = D_{ijk}(x,y) n_j(y) \tag{7}$$

Conversely:

$$\Sigma_{ij}(e_k, x, y) = D_{ijk}(x,y) = T_{ik}(x, y, n_y = e_j) \tag{8}$$

Because tensors Σ and T are functions of D, any equation in the sequel can be expressed in terms of D alone. In practice, however, the simultaneous use of notations T and D proves to be more convenient.

The equilibrium equation in terms of the fundamental solution reads

$$\begin{aligned}&\forall k, \quad \operatorname{div}\Sigma(e_k, x, y) + \delta(y-x)\cdot e_k = 0, \quad \text{i.e.,} \\ &\forall i, \quad k, \frac{\partial D_{ijk}}{\partial y_j}(x,y) + \delta(y-x)\cdot \delta_{ik} = 0\end{aligned} \tag{9}$$

which gives rise to the so-called rigid body identity:

$$\forall x \in \Omega \setminus S, \quad \int_S D_{mnk}(x,y) n_n(y)\, d_y S = -\delta_{mk} \tag{10}$$

where Ω is any bounded region and S its boundary.

We now come to the expression of the fundamental solution and its derivatives. For this purpose, let us introduce the following notations. Given two points x and y of space $\mathscr{E}$, let (a_1, a_2, a) be an orthogonal base with the third vector $\mathbf{a}$ equal to $e_r = (y-x)/\|y-x\|$, a_1 and a_2 being arbitrary. This based allows us to define the spherical coordinates $(\chi, \psi) \in [0, \pi[\times [0, 2\pi[$ such that the line $\chi = 0$ coincides with e_r, whereas the origin for ψ is arbitrary.

Let us define the tensor Q as $Q_{ik}(\zeta) = C_{ijkl}\zeta_j\zeta_l$, ζ_j is the j-component of vector $\boldsymbol{\zeta}$ in the global fixed base (e_1, e_2, e_3). The inverse of Q is denoted by P:

$$\begin{aligned}&P(\zeta) = Q^{-1}(\zeta), \quad \text{i.e.,} \\ &P_{ik}(\zeta) = \frac{1}{2\det Q}\varepsilon_{kpq}\varepsilon_{irs} Q_{pr} Q_{qs}\end{aligned}$$

where ε_{kpq} is the permutation symbol. Both P and Q are symmetric tensors. Since a unit vector $\boldsymbol{\zeta}$ can be entirely determined by its spherical coordinates (χ, ψ) in the base (a_1, a_2, a), tensors $P(\zeta)$ and $Q(\zeta)$ are also written as $P(a, \chi, \psi)$ and $Q(a, \chi, \psi)$.

The following theorem was proved by Vogel and Rizzo[16] by decomposing the Dirac function into plane waves.

Theorem

The fundamental solution for an anisotropic elastic medium is given by:

$$\forall m, \quad U(e_m, x, y) = \frac{1}{8\pi^2 r}\int_0^{2\pi} P(a,\psi)\,d\psi\cdot e_m$$

$$\Leftrightarrow E(x,y) = \frac{1}{8\pi^2 r}\int_0^{2\pi} P(a,\psi)\,d\psi \qquad (11)$$

in which $r = \|y - x\|$ and $P(a,\psi)$ stands for $P(a, \chi = \pi/2, \psi)$.

The integral is taken along the unit circle in the plane normal to $a = e_r$ and passing through x.

The derivatives of tensor $E(x,y)$ can be computed by

$$\forall k,p,q,$$

$$\frac{\partial E_{pk}}{\partial y_q}(x,y) = -\frac{1}{8\pi^2 r^2}\left\{ r_{,q}\int_0^{2\pi} P_{pk}(a,\psi)\,d\psi + (\delta_{nq} - r_{,n}r_{,q})\int_0^{2\pi} P_{pl}(a,\psi) \times \frac{\partial Q_{lm}}{\partial a_n}(a,\psi)P_{mk}(a,\psi)\,d\psi \right\} \qquad (12)$$

where $r_{,i} = dr/dy_i = e_r\cdot e_i$ and $Q(a,\psi)$ stands for $Q(a, \chi = \pi/2, \psi)$.

The following properties result from the above theorem.

- The symmetry of E:

$$E(x,y) = E^T(x,y) \qquad (13)$$

- By applying the same reasoning made for $E(x,y)$ to $E(y,x)$, we obtain the variable interchange properties:

$$E(x,y) = E(y,x) \qquad (14)$$

$$\frac{\partial E}{\partial x_l}(x,y) = -\frac{\partial E}{\partial y_l}(x,y) \qquad (15)$$

- Using relation (5) and the foregoing properties, we obtain similar relations for Σ and D:

$$\Sigma(e_k,x,y) = -\Sigma(e_k,y,x) \quad \text{and} \quad D(x,y) = -D(y,x) \qquad (16)$$

$$\frac{\partial \Sigma}{\partial x_l}(e_k,x,y) = -\frac{\partial \Sigma}{\partial y_l}(e_k,x,y) \quad \text{and} \quad \frac{\partial D}{\partial x_l}(x,y) = -\frac{\partial D}{\partial y_l}(x,y) \qquad (17)$$

- The singularity of the fundamental solution:

$$E(x,y) = O\left(\frac{1}{r}\right) \qquad D(x,y) = O\left(\frac{1}{r^2}\right)$$

$$T(x,y,n_y) = O\left(\frac{1}{r^2}\right) \qquad \text{as } r = \|y-x\| \to 0 \qquad (18)$$

- The asymptotic behavior at infinity:

$$E(x,y) = O\left(\frac{1}{r}\right) \qquad D(x,y) = O\left(\frac{1}{r^2}\right)$$

$$T(x,y,n_y) = O\left(\frac{1}{r^2}\right) \qquad \text{as } r = \|y-x\| \to \infty \qquad (19)$$

Of course, in isotropic or transversely isotropic cases, relations (11) and (12) simplify appreciably, yielding well-known expressions.[21] The fundamental solution for a general anisotropic medium is known only by the integral representation (11), but the expression in closed form is not available in general since the actual integration is not possible for any time of anisotropy. Nevertheless, the above properties prove to be sufficient to establish the so-called limit theorems in what follows, without the knowledge of any closed form expressions for E and T whatever.

Limit theorems

The purpose of this section is to present some results about the limit behavior of the fundamental solution when the load point x approaches a point y_0 belonging to a given surface S. These results hold in the case of general anisotropy, thus constituting the generalization of the well-known ones in the isotropic case. They all are based on the lemma below, which alone exploits the integral representation (12) of the fundamental solution.

First let us introduce some notations. Given a point y_0 and a unit vector $\mathbf{n}_{y0}$, let Π be the plane passing through y_0 and perpendicular to $\mathbf{n}_{y0}$. The normal $\mathbf{n}_{y0}$ defines two sides of plane Π, which will be recognized by the signs $+$ and $-$, the $+$ side being the half-space containing the point $y_0 + \mathbf{n}_{y0}$.

Let ε be an arbitrary positive number, $B(y_0,\varepsilon)$ the ball centered at y_0, and with radius ε. The plane Π divides the boundary $\partial B(y_0,\varepsilon)$ of the ball into two parts denoted by $S_1(y_0,\varepsilon)$ and $S_2(y_0,\varepsilon)$, situated in the sides $+$ and $-$, respectively.

Lemma

$$\forall y_0, \forall \mathbf{n}_{y0}, \forall \varepsilon, \forall k,l,p,q,$$

$$\int_{S_1(y_0,\varepsilon)} \frac{\partial E_{pk}}{\partial y_q}(y_0,y)n_l(y)\,d_yS = \int_{S_2(y_0,\varepsilon)} \frac{\partial E_{pk}}{\partial y_q}(y_0,y)n_l(y)\,d_yS \qquad (20)$$

where the normal n_y is outward to $B(y_0,\varepsilon)$.

Proof:

Regarding the integral over $S_2(y_0, \varepsilon)$, relation (12) yields:

$$\int_{S_2(y_0,\varepsilon)} \frac{\partial E_{pk}}{\partial y_q}(y_0, y) n_l(y)\, d_y S$$

$$= -\frac{1}{8\pi^2}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} r_{,l}\left\{ r_{,q}\int_{\psi=0}^{2\pi} P_{pk}(a,\psi)\, d\psi \right.$$

$$+ (\delta_{nq} - r_{,n}r_{,q})\int_{\psi=0}^{2\pi} P_{pl}(a,\psi)\frac{\partial Q_{lm}}{\partial a_n}(a,\psi)$$

$$\left. \times P_{mk}(a,\psi)\, d\psi \right\} \sin\theta\, d\theta\, d\varphi \qquad (21)$$

in which $r = \|y - y_0\|$, $r_{,q} = (y_q - y_{0q})/r$, $e_r = (y - y_0)/r$, the spherical coordinates $(\theta, \varphi) \in [0, \pi[x[0, 2\pi[$ are so defined that the line $\theta = 0$ coincides with n_{y0} whereas the origin for φ is arbitrary. As for the integral over $S_1(y_0, \varepsilon)$, it is identical to that of (21), only the bounds for θ are different, since now θ varies from $\pi/2$ to π. Using the variable change $\theta' = \pi - \theta$, $\varphi' = \pi - \varphi$ and introducing new notations: y' denoting the symmetrical point of y with respect to y_0, $r' = \|y' - y_0\|$, $r'_{,q} = \partial r'/\partial y'_q = (y'_q - y_{0q})/r'$ and $a' = -a$, we have:

$$\int_{S_1(y_0,\varepsilon)} \frac{\partial E_{pk}}{\partial y_q}(y_0, y) n_l(y)\, d_y S$$

$$= -\frac{1}{8\pi^2}\int_{\theta'=\pi/2}^{0}\int_{\varphi'=\pi}^{-\pi}(-r'_{,l})$$

$$\times\left\{ (-r'_{,q})\int_{\psi=0}^{2\pi} P_{pk}(-a',\psi)\, d\psi \right.$$

$$+ (\delta_{nq} - r'_{,n}r'_{,q})\int_{\psi=0}^{2\pi} P_{pl}(-a',\psi)\frac{\partial Q_{lm}}{\partial(-a'_n)}$$

$$\left. \times(-a',\psi) P_{mk}(-a',\psi)\, d\psi \right\} \sin\theta'\, d\theta'\, d\varphi' \qquad (22)$$

Eventually, by comparing relations (21) and (22), and by noting that:

$$\int_0^{2\pi} P_{pk}(-a',\psi)\, d\psi = \int_0^{2\pi} P_{pk}(a',\psi)\, d\psi$$

we obtain the proof of the lemma.

Applying relations (5) and (6) to relation (20), we directly deduce the following propositions.

Proposition

$$\forall y_0, \forall \mathbf{n}_{y0}, \forall\varepsilon, \forall i, k,$$

$$\int_{S_1(y_0,\varepsilon)} T_{ik}(y_0, y, \mathbf{n}_y)\, \mathbf{d}_y S$$

$$= \int_{S_2(y_0,\varepsilon)} T_{ik}(y_0, y, \mathbf{n}_y)\, d_y S \qquad (23)$$

Proposition

$$\forall y_0, \forall \mathbf{n}_{y0}, \forall\varepsilon, \forall i, j, k, l,$$

$$\int_{S_1(y_0,\varepsilon)} D_{ijk}(y_0, y) n_l(y)\, d_y S$$

$$= \int_{S_2(y_0,\varepsilon)} D_{ijk}(y_0, y) n_l(y)\, d_y S \qquad (24)$$

It should be noted that the above propositions involve the plane Π passing through y_0 and normal to vector $\mathbf{n}_{y0}$. Moreover, the value of ε is arbitrary, not necessarily small. In the subsequent applications where y_0 belongs to an arbitrarily shaped surface S, the neighborhood of y_0 *in* S is *not* plane, this is why we shall have to take the limit $\varepsilon \to 0$.

Relations (23) and (24) allow us to prove the so-called limit theorems below. Let us first agree about notations for the orientation of a surface. Of course, any surface S (closed or open) considered here is assumed to be orientable. This implies that, for any point $y_0 \in S \setminus \partial S$, we can locally define two sides of S which we label side $+$ and side $-$, all the normals to S being directed from side $-$ to side $+$.

Theorem

Let S be a surface (closed or open) and $\mathbf{u}$ a vector field defined on S. If:

(i) S is a Lyapunov surface: $S \in C^{1,\alpha}$, $0 < \alpha \le 1$, which means that

$$\exists C > 0, \forall y, y' \in S, \quad \|\mathbf{n}_{y'} - \mathbf{n}_y\| \le C\|y' - y\|^\alpha$$

(ii) $\mathbf{u}$ satisfies the Holder condition on S: $\mathbf{u} \in C^{0,\beta}(S)$, $0 < \beta \le 1$, i.e.,

$$\exists C > 0, \forall x, y \in S, \quad \|u(y) - u(x)\| \le C\|y - x\|^\beta$$

then

$$\forall y_0 \in S$$

$$\lim_{x \to y_0^\pm} \int_S T(x, y, \mathbf{n}_y) u(y)\, d_y S$$

$$= \pm\frac{1}{2}u(y_0) + pv\int_S T(y_0, y, \mathbf{n}_y) u(y)\, d_y S \qquad (25)$$

where by $x \to y_0^\pm$ are meant the limits as x approaches y_0, x belonging to the side $+$ and the side $-$, respectively. The symbol pv denotes a Cauchy principal value integral. Relation (25) also holds if tensor T is replaced by its transpose T^T.

Proof:

By denoting $S(y_0, \varepsilon) = S \cap B(y_0, \varepsilon)$, the left-hand side of (25) can be recast as

$$\lim_{x \to y_0^\pm} \int_S T(x, y, \mathbf{n}_y) u(y)\, d_y S$$

$$= \lim_{x \to y_0^\pm} \int_{S \setminus S(y_0, \varepsilon)} T(x, y, \mathbf{n}_y) u(y)\, d_y S$$

$$+ \lim_{x \to y_0^\pm} \int_{S(y_0, \varepsilon)} T(x, y, \mathbf{n}_y)[u(y) - u(y_0)]\, d_y S$$

$$+ \lim_{x \to y_0^\pm} \int_{S(y_0, \varepsilon)} T(x, y, \mathbf{n}_y)\, d_y S \cdot u(y_0) \qquad (26)$$

- The first integral is continuous at y_0 when $y \to y_0$. It tends to the Cauchy principal value integral appearing in the right-hand side of (25) as $\varepsilon \to 0$.
- Let us denote $r = \|y - x\|$, $r_0 = \|y_0 - x\|$ and $\rho = \|y - y_0\|$. From (18) and hypothesis (ii), we have: $T(x, y, \mathbf{n}_y) = O(1/r^2) = O[1/(\rho^2 + r_0^2)]$, $u(y) - u(y_0) = O(\rho^\beta)$, and $d_y S = O(\rho)$. Hence:

$$\lim_{x \to y_0^\pm} \int_{S(y_0, \varepsilon)} T(x, y, \mathbf{n}_y)[u(y) - u(y_0)]\, d_y S = O(\varepsilon^\beta)$$

- It remains to investigate the last integral in equation (26). For brevity, let us denote:

$$A^\pm = \lim_{x \to y_0^\pm} \int_{S(y_0, \varepsilon)} T(x, y, \mathbf{n}_y)\, d_y S$$

$$B = \lim_{x \to y_0^\pm} \int_{S_1(y_0, \varepsilon)} T(x, y, \mathbf{n}_y)\, d_y S$$

$$= \int_{S_1(y_0, \varepsilon)} T(y_0, y, \mathbf{n}_y)\, d_y S$$

$$C = \lim_{x \to y_0^\pm} \int_{S_2(y_0, \varepsilon)} T(x, y, \mathbf{n}_y)\, d_y S$$

$$= \int_{S_2(y_0, \varepsilon)} T(y_0, y, \mathbf{n}_y)\, d_y S$$

where the normal $\mathbf{n}_y$ for $y \in S_1(y_0, \varepsilon) \cup S_2(y_0, \varepsilon)$ is outward to the ball $B(y_0, \varepsilon)$ while we recall that $\mathbf{n}_y$ for $y \in S(y_0, \varepsilon)$ is directed from side $-$ to side $+$.

The rigid body identity (10) yields:

$$A^+ + B = 0, \quad A^- + B = -I \qquad (27)$$

where I is the unit tensor. Similarly, we have:

$$-A^+ + C = -I, \quad -A^- + C = 0 \qquad (28)$$

On the other hand, from hypothesis (i), $S_1(y_0, \varepsilon)$ and $S_2(y_0, \varepsilon)$ tend to two symmetrical hemispheres as $\varepsilon \to 0$. Therefore, by applying equation (23), we get

$$\lim_{\varepsilon \to 0} B = \lim_{\varepsilon \to 0} C \qquad (29)$$

Solving equations (27) to (29) gives

$$\lim_{\varepsilon \to 0} A^\pm = \pm \frac{I}{2}.$$

The theorem is proved.

Theorem

Let S be a surface (closed or open) and $\mathbf{t}$ a vector field defined on S. If:

(i) $S \in C^{1,\alpha}$, $0 < \alpha \le 1$
(ii) $\mathbf{t} \in C^{0,\beta}(S)$, $0 < \beta \le 1$
then

$$\forall y_0 \in S$$

$$\lim_{x \to y_0^\pm} \int_S D(y, x)\mathbf{t}(y, \mathbf{n}_y)\, d_y S \cdot \mathbf{n}_{y0}$$

$$= \mp \frac{1}{2} \mathbf{t}(y_0, \mathbf{n}_{y0})$$

$$+ pv \int_S D(y, y_0)\mathbf{t}(y, \mathbf{n}_y)\, d_y S \cdot \mathbf{n}_{y0} \qquad (30)$$

where $\mathbf{n}_{y0}$ is the normal vector at y_0 to S and the product of the third-order tensor D and vector $\mathbf{t}$ is the second-order tensor defined by $(D \cdot \mathbf{t})_{ij} = D_{ijm}\mathbf{t}_m$.

Proof:

The proof is similar to that of equation (25), bearing in mind that, in view of relations (7) and (16):

$$\left[D(y, x)\mathbf{t}(y_0, \mathbf{x}_{y0})\right] \cdot \mathbf{n}_y = -T(x, y, \mathbf{n}_y)\mathbf{t}(y_0, \mathbf{n}_{y0})$$

The following theorem requires a somewhat stronger condition for the function u.

Theorem

Assuming that

(i) $S \in C^{1,\alpha}$, $0 < \alpha \le 1$
(ii) $u \in C^{1,\beta}(S)$, $0 < \beta \le 1$, i.e., all the derivatives of u belong to the class $C^{0,\beta}(S)$ we have the property of continuity across the boundary:

$$\forall y_0 \in S$$

$$\lim_{x \to y_0^\pm} \int_S \mathscr{R}(\partial y, x, y, \mathbf{n}_y) u(y)\, d_y S \cdot \mathbf{n}_{y0}$$

$$= pv \int_S \mathscr{R}(\partial y, y_0, y, \mathbf{n}_y) u(y)\, d_y S \cdot \mathbf{n}_{y0} \qquad (31)$$

where the symbol $\mathscr{R}$ represents the differential operator defined as:

$$\left[\mathscr{R}(\partial y, x, y, \mathbf{n}_y)u(y)\right]_{ij}$$

$$=\mathscr{R}_{ijm}(\partial y, x, y, \mathbf{n}_y)u_m(y)$$

$$= C_{ijkl}D_{mnk}(x,y)\cdot\mathscr{D}_{nl}(\partial y, \mathbf{n}_y)u_m(y) \tag{32}$$

$\mathscr{D}_{nl}(\partial y, \mathbf{n}_y)$ is the tangential differential operator defined as

$$\mathscr{D}_{nl}(\partial y, \mathbf{n}_y) = n_n(y)\frac{\partial}{\partial y_l} - n_l(y)\frac{\partial}{\partial y_n} \tag{33}$$

(there is no possible confusion of the normal $\mathbf{n}_y$ with the subscript $n \in \{1,2,3\}$), the symbol ∂y recalls that the differentiation is carried out with respect to variable y.

Proof:
Invoking arguments similar to those in the proof of (25), we can write:

$$\lim_{x\to y_0^\pm}\int_S \mathscr{R}(\partial y, x, y, \mathbf{n}_y)u(y)\, d_yS\cdot\mathbf{n}_{y0}$$

$$= \lim_{\varepsilon\to 0}\lim_{x\to y_0^\pm}\int_{S(y_0,\varepsilon)}\mathscr{R}(\partial y, x, y, \mathbf{n}_y)u(y_0)\, d_yS$$

$$\cdot\mathbf{n}_{y0} + pv\int_S \mathscr{R}(\partial y, y_0, y, \mathbf{n}_y)u(y)\, d_yS\cdot\mathbf{n}_{y0}$$

The main work is the study of

$$A^\pm = \lim_{x\to y_0^\pm}\int_{S(y_0,\varepsilon)}\mathscr{R}_{ijm}(\partial y, x, y, \mathbf{n}_y)u_m(y_0)\, d_yS\cdot\mathbf{n}_{j0}$$

$$\left[\mathbf{n}_{j0} \equiv n_j(y_0)\right]$$

Let us also denote

$$B = \lim_{x\to y_0^\pm}\int_{S_1(y_0,\varepsilon)}\mathscr{R}_{ijm}(\partial y, x, y, \mathbf{n}_y)u_m(y_0)\, d_yS\cdot\mathbf{n}_{j0}$$

$$= \int_{S_1(y_0,\varepsilon)}\mathscr{R}_{ijm}(\partial y, y_0, y, \mathbf{n}_y)u_m(y_0)\, d_yS\cdot\mathbf{n}_{j0}$$

$$C = \lim_{x\to y_0^\pm}\int_{S_2(y_0,\varepsilon)}\mathscr{R}_{ijm}(\partial y, x, y, \mathbf{n}_y)u_m(y_0)\, d_yS\cdot\mathbf{n}_{j0}$$

$$= \int_{S_2(y_0,\varepsilon)}\mathscr{R}_{ijm}(\partial y, y_0, y, \mathbf{n}_y)u_m(y_0)\, d_yS\cdot\mathbf{n}_{j0}$$

where the normal $\mathbf{n}_y$ on different surfaces is defined as in the proof of (25).

- First let $S_1(y_0, \varepsilon)$ be involved by writing

$$A^\pm + B = \lim_{x\to y_0^\pm} C_{ijkl}\int_{S(y_0,\varepsilon)\cup S_1(y_0,\varepsilon)} D_{mnk}(x,y)$$

$$\times[n_n u_{m,l}(y_0) - n_l u_{m,n}(y_0)]\, d_yS\cdot\mathbf{n}_{j0} \tag{34}$$

where $n_l = n_l(y)$. For the sake of brevity, the right-hand side of the previous equation will be written in the form of a difference

$$A^\pm + B = B^\pm - G^\pm \tag{35}$$

In virtue of the rigid body identity (10), we have

$$F^+ = 0,\ F^- = -C_{ijkl}u_{k,l}(y_0)\mathbf{n}_{j0} \tag{36}$$

Moreover, $G^\pm$ can be transformed successively as follows. Using equation (5),

$$G^\pm = \lim_{x\to y_0^\pm} C_{ijkl}C_{mnpq}\int_{S(y_0,\varepsilon)\cup S_1(y_0,\varepsilon)}\frac{\partial E_{pk}}{\partial y_q}(x,y)$$

$$\times n_l\, d_yS\cdot\mathbf{u}_{m,n}(y_0)\mathbf{n}_{j0}$$

Now, since $S(y_0, \varepsilon)\cup S_1(y_0, \varepsilon)$ is a closed surface, we have

$$\int_{S(y_0,\varepsilon)\cup S_1(y_0,\varepsilon)}\frac{\partial E_{pk}}{\partial y_q}n_l\, d_yS$$

$$= \int_{S(y_0,\varepsilon)\cup S_1(y_0,\varepsilon)}\frac{\partial E_{pk}}{\partial y_l}n_q\, d_yS$$

Then, owing to the symmetry of E, relation (14), and relation (5):

$$G^\pm = \lim_{x\to y_0^\pm} C_{mnpq}\int_{S(y_0,\varepsilon)\cup S_1(y_0,\varepsilon)}$$

$$\times D_{ijp}(x,y)n_q\, d_yS\cdot\mathbf{u}_{m,n}(y_0)n_{j0}$$

$$= \lim_{x\to y_0^\pm} C_{mnpq}\int_{S(y_0,\varepsilon)\cup S_1(y_0,\varepsilon)}$$

$$\times D_{ijp}(n_{j0}n_q - \mathbf{n}_j n_{q0})\, d_yS\cdot\mathbf{u}_{m,n}(y_0)$$

$$+ \lim_{x\to y_0^\pm} C_{mnpq}\int_{S(y_0,\varepsilon)\cup S_1(y_0,\varepsilon)}$$

$$\times D_{ijp}n_j\, d_yS\cdot\mathbf{u}_{m,n}(y_0)n_{q0}$$

Using equation (10) and the symmetry $C_{mnij} = C_{ijmn}$ relation (1), we get

$$G^+ = \lim_{x\to y_0^\pm} C_{mnpq}\int_{S(y_0,\varepsilon)\cup S_1(y_0,\varepsilon)}$$

$$\times D_{ijp}(n_{j0}n_q - n_j n_{q0})\, d_yS\cdot\mathbf{u}_{m,n}(y_0) \tag{37}$$

and

$$G^- = \lim_{x \to y_0^\pm} C_{mnpq} \int_{S(y_0,\varepsilon) \cup S_1(y_0,\varepsilon)} \times D_{ijp}(n_{j0} n_q - n_j n_{q0})\, d_y S \cdot \mathbf{u}_{m,n}(y_0) - C_{ijmn} u_{m,n}(y_0) n_{j0}$$

Then, substituting equations (36) and (37) into (35) gives:

$$A^\pm + B = \lim_{x \to x_0^\pm} C_{mnpq} \int_{S(y_0,\varepsilon) \cup S_1(y_0,\varepsilon)} \times D_{ijp}(x,y)(-n_{j0} n_q + n_j n_{q0})\, d_y S \cdot \mathbf{u}_{m,n}(y_0)$$
$$= D + \lim_{x \to y_0^\pm} C_{mnpq} \int_{S(y_0,\varepsilon)} D_{ijp}(x,y) \times (-n_{j0} n_q + n_j n_{q0})\, d_y S \cdot \mathbf{u}_{m,n}(y_0)$$

where

$$D = C_{mnpq} \int_{S_1(y_0,\varepsilon)} D_{ijp}(y_0, y) \times (-n_{j0} n_q + n_j n_{q0})\, d_y S \cdot \mathbf{u}_{m,n}(y_0)$$

But, from equation (18) and hypothesis (i):

$$\int_{S(y_0,\varepsilon)} D_{ijp}(x,y)(-n_{j0} n_q + n_j n_{q0})\, d_y S = \int_{S(y_0,\varepsilon)} D_{ijp}(x,y)\big[(n_j - n_{j0}) n_{q0} - n_{j0}(n_q - n_{q0})\big]\, d_y S = O(\varepsilon^\alpha)$$

Thus,

$$A^\pm + B = D + O(\varepsilon^\alpha) \tag{38}$$

- The same treatment can be applied to $S_2(y_0, \varepsilon)$, with special attention to the normal orientation. The corresponding results for the $\pm$ cases are interchanged with respect to those obtained with $S_1(y_0, \varepsilon)$, and we are led to

$$-A^\pm + C = E + O(\varepsilon^\alpha) \tag{39}$$

where

$$E = C_{mnpq} \int_{S_2(y_0,\varepsilon)} D_{ijp}(y_0, y) \times (-n_{j0} n_q + n_j n_{q0})\, d_y S \cdot \mathbf{u}_{m,n}(y_0)$$

- Furthermore, according to hypothesis (i), $S_1(y_0, \varepsilon)$ and $S_2(y_0, \varepsilon)$ tend to two symmetrical hemispheres as $\varepsilon \to 0$, so that equation (24) holds, yielding

$$\lim_{\varepsilon \to 0} B = \lim_{\varepsilon \to 0} C \tag{40}$$

and

$$\lim_{\varepsilon \to 0} D = \lim_{\varepsilon \to 0} E \tag{41}$$

- Eventually, relations (38)–(41) form an *underdetermined* algebraic system of four equations for five unknowns, A, B, C, D, and E, from which we derive a *definite* result for A: $\lim_{\varepsilon \to 0} A^\pm = 0$.

The theorem is proved.

Notes

(a) Relations (25), (30), and (31), established in the anisotropic case, constitute the generalization of well-known results in isotropy where they can be directly verified using the closed form expressions available for T, D, and $\mathscr{R}$ (Ref. 19, pp. 29 and 43).
(b) The hypothesis (ii) for equation (31), $u \in C^{1,\beta}(S)$, is stronger than that for equation (25) because of the derivatives involved by the differential operator $\mathscr{R}$.

3. Application to fracture mechanics: Integral representation of the displacement in a cracked body

Consider a linear elastic anisotropic, finite or infinite, body Ω containing a crack. If the body is finite, its outer boundary is denoted S_B and the crack can be either an embedded one or a surface one. The crack surface S_{cr} is made up of two faces S_{cr}^+ and S_{cr}^- which coincide in the undeformed state. To each point $y \in S_{cr}$ correspond two points y^+ and y^- belonging, respectively, to S_{cr}^+ and S_{cr}^-. The respective normal vectors are opposite, i.e., $\mathbf{n}_y^+ = \mathbf{n}_y^-$, $\mathbf{n}_y^-$ being directed from S_{cr}^- to S_{cr}^+, thus defined everywhere as outward with respect to the body considered, in accordance with the usual convention. In the sequel, all the equations will be written using S_{cr}^-, so that normal $\mathbf{n}_y^-$ is taken as the reference one.

This section gives the displacement at any interior point of the body in terms of boundary quantities. Two cases are considered: the infinite body (with an embedded crack) and a finite body with an embedded or surface crack.

3.1 *Infinite body*

Consider the infinite body Ω containing the crack $S_{cr} = S_{cr}^- \cup S_{cr}^+$. The following hypotheses referred to as the regularity conditions are assumed.

Regularity conditions

(i) The radiation condition for unknown displacement and stress is assumed as: $u(y) = o(1)$, $\sigma(y) = o(1/r)$, i.e., $t(y,n_y)=o(1/r)$, when $r=\|y-x\|\to\infty$, x being a fixed point.

(ii) The radiation condition for the body forces is assumed as: $f(y)=O(r^{-2-\delta})$ when $r\to\infty$, where δ is a positive constant smaller than 1.

Note that condition (ii) is identically verified if the body forces f is confined to a finite region. Given any point x, let $B(x,R)$ denote the ball centered at x with radius R large enough for $B(x,R)$ to include the crack. Let S_R be the exterior surface of $B(x,R)$ and Ω_R, be the *bounded* region within $S_{cr}^-\cup S_{cr}^+$ and S_R, with boundary $S_{cr}^-\cup S_{cr}^+\cup S_R$. The second condition (ii) ensures that $\lim_{R\to\infty}\int_{\Omega_R}E(x,y)f(y)\,d_yV$ is bounded.

The following theorem holds.

Theorem

Provided the regularity conditions are fulfilled, we can write the integral representation of the displacement as:

$$\begin{aligned}
&\forall x\notin S_{cr},\\
&u(x)=\int_{S_{cr}^-}\big[E(x,y)\Sigma\mathbf{t}(y)\\
&\qquad+T^T(x,y,\mathbf{n}_y)\,\Delta u(y)\big]\,d_yS\\
&\qquad+*\int_\Omega E(x,y)f(y)\,d_yV
\end{aligned}\tag{42}$$

in which $\Sigma\mathbf{t}(y)$ is the sum of the stress vectors on the crack faces: $\Sigma\mathbf{t}(y)=\mathbf{t}(y^+,\mathbf{n}_y^+)+\mathbf{t}(y^-,\mathbf{n}_y^-)$ (if the crack is loaded symmetrically, i.e., $\mathbf{t}(y^+,\mathbf{n}_y^+)=-\mathbf{t}(\mathbf{y}^-,\mathbf{n}_y^-)$, then $\Sigma\mathbf{t}(y)=0$) Δu is the displacement jump through the crack: $\Delta u(y)=u(y^+)-u(y^-)$, and the asterisk denotes an improper integral.

The proof of the theorem does not formally differ from that in the isotropic case.

3.2 Finite body

Consider a finite body Ω with outer boundary S_B, containing a crack S_{cr}. The following theorem can be proved in a similar way as in the infinite body case.

Theorem

The integral representation of the displacement reads:

$$\begin{aligned}
&\forall x\in\Omega\setminus(S_B\cup S_{cr}),\\
&u(x)=\int_{S_{cr}^-}\big[E(x,y)\Sigma\mathbf{t}(y)\\
&\qquad+T^T(x,y,\mathbf{n}_y)\,\Delta u(y)\big]\,d_yS\\
&\qquad+\int_{S_B}\big[E(x,y)\mathbf{t}(y,\mathbf{n}_y)\\
&\qquad\quad-T^T(x,y,\mathbf{n}_y)u(y)\big]\,d_yS\\
&\qquad+*\int_\Omega E(x,y)f(y)\,d_yV
\end{aligned}\tag{43}$$

Note that the regularity conditions are useless in the finite body case.

4. Integral representation of the stress

In practical purposes, it is often necessary to compute the complete stress tensor at any point inside the body. This section is thus devoted to the integral representation of the internal stress.

4.1 Infinite body

Theorem

Assuming that:

(i) the regularity conditions are fulfilled.

(ii) $S_{cr}^-\in C^{1,\alpha}$, $0<\alpha\le 1$

(iii) $\Delta u\in C^1(S_{cr}^-)$

we have the integral representation of the stress:

$$\begin{aligned}
\forall x\notin S_{cr},\quad \sigma(x)=&\int_{S_{cr}^-}\{D(y,x)\Sigma\mathbf{t}(y)\\
&+\mathscr{R}(\partial y,x,y,\mathbf{n}_y)\,\Delta u(y)\}\,d_yS\\
&+*\int_\Omega D(y,x)f(y)\,d_yV
\end{aligned}\tag{44}$$

Proof:

The integral representation of the displacement (42) holds because of hypothesis (i). Given a point $x\notin S_{cr}$, the derivative $\partial u_k(x)/\partial x_l$ will be investigated to obtain the stress $\sigma(x)$. All integrals in equation (42) can be differentiated behind the integral sign. Indeed, since x is interior to Ω the surface integral is regular. Moreover, relation (18) implies that the kernel of the volume integral behaves as $1/r$ as $r\to 0$. Thus:

$$\begin{aligned}
\frac{\partial u_k}{\partial x_l}(x)=&\int_{S_{cr}^-}\left\{\frac{\partial E_{mk}}{\partial x_l}(x,y)\Sigma t_m(y,\mathbf{n}_y)\right.\\
&\left.+\frac{\partial T_{mk}}{\partial x_l}(x,y,\mathbf{n}_y)\,\Delta u_m(y)\right\}d_yS\\
&+*\int_\Omega\frac{\partial E_{mk}}{\partial x_l}(x,y)f_m(y)\,d_yV
\end{aligned}\tag{45}$$

The first and third integrals containing $\partial E_{mk}/\partial x_l$ do not require any further transformations. As regards the second integral, exploiting hypotheses (ii) and (iii) it can be transformed by means of the so-called regularization theorem (see Appendix):

$$\int_{S_{cr}^-} \frac{\partial T_{mk}}{\partial x_l}(x, y, \mathbf{n}_y)\, \Delta u_m(y)\, d_y S = \int_{S_{cr}^-} D_{mnk}(x, y) \mathscr{D}_{nl}(\partial y, n_y)\, \Delta u_m(y)\, d_y S \tag{46}$$

In writing equation (46), use has been made of the boundary condition along the crack front: $\Delta u(y) = 0$ for $y \in \partial S_{cr}$. The theorem is proved.

Note the order of variables x and y in the D-terms of equation (44): $D(y, x)$ instead of $D(x, y)$. On the other hand, the volume integral in equation (44) is improper convergent on account of the regularity condition $f \in O(r^{-2-\delta})$.

4.2 Finite body

Theorem

Assuming that:

(i) $S_{cr}^- \in C^{1,\alpha}$, $S_B \in C^{1,\alpha'}$, $0 < \alpha, \alpha' \leq 1$
(ii) $\Delta u \in C^1(S_{cr}^-)$, $u \in C^1(S_B)$
we have the integral representation of the stress for a finite cracked body:

$$\forall x \in \Omega \setminus (S_B \cup S_{cr}),$$

$$\begin{aligned}\sigma(x) = &\int_{S_{cr}^-} \{D(y, x) \Sigma \mathbf{t}(y) \\ &+ \mathscr{R}(\partial y, x, y, \mathbf{n}_y)\, \Delta u(y)\}\, d_y S \\ &+ \int_{S_B} \{D(y, x) \mathbf{t}(y, \mathbf{n}_y) \\ &- \mathscr{R}(\partial y, x, y, \mathbf{n}_y) u(y)\}\, d_y S \\ &+ * \int_{\Omega} D(y, x) f(y)\, d_y V \end{aligned} \tag{47}$$

In the case of a surface crack, S_B must be replaced by $S_B \setminus L$ throughout, where $L = \partial S_{cr}^- \cap S_B$ is referred to as the surface line.

Proof:

Only the case of the surface crack, which needs special attention, is investigated; the proof for the embedded crack can be deduced in an obvious way. The reasoning is essentially the same as in the infinite body case, the difference being in that here there are two surfaces: the closed surface S_B upon which the displacement u and the stress vector $\mathbf{t}$ are defined and the *open* surface S_{cr}^- upon which the displacement jump Δu and the stress sum $\Sigma \mathbf{t}$ are defined.

The surface line L being positively oriented with respect to the normal $\mathbf{n}_y^-$ of S_{cr}^-, let us introduce the closed contour $L^- \cup L^+$ made up to two arcs L^- and L^+ such that both of them coincide with L whereas the orientation of L^- (respectively L^+) is the opposite to (respectively the same as) that of L (*Figure 1*).

Assume for definiteness that the contour $L^- \cup L^+$ is oriented negatively with respect to the outward normal to the exterior boundary S_B, as shown in *Figure 1*. In fact, it can be easily verified that the final result does not actually depend on this assumption.

Hypotheses (i) and (ii) make it possible to apply the regularization theorem (see Appendix) successively to surfaces S_{cr}^- and $S_B \setminus L$ the boundary of which is $L^- \cup L^+$:

$$\begin{aligned}\frac{\partial}{\partial x_l} &\int_{S_{cr}^-} T^T(x, y, \mathbf{n}_y)\, \Delta u(y)\, d_y S \\ &= \text{surface integral over } S_{cr}^- \\ &\quad - \int_L \varepsilon_{nlr}\, \Delta u_m(y)\, D_{mnk}(x, y)\, dy_r \end{aligned} \tag{48}$$

and

$$\begin{aligned}- \frac{\partial}{\partial x_l} &\int_{S_B} T^T(x, y, \mathbf{n}_y) u(y)\, d_y S \\ &= - \frac{\partial}{\partial x_l} \int_{S_B \setminus L} T^T(x, y, \mathbf{n}_y) u(y)\, d_y S \\ &= \text{surface integral over } S_B \setminus L \\ &\quad + \int_{L^- \cup L^+} \varepsilon_{nlr} u_m(y) D_{mnk}(x, y)\, dy_r \end{aligned} \tag{49}$$

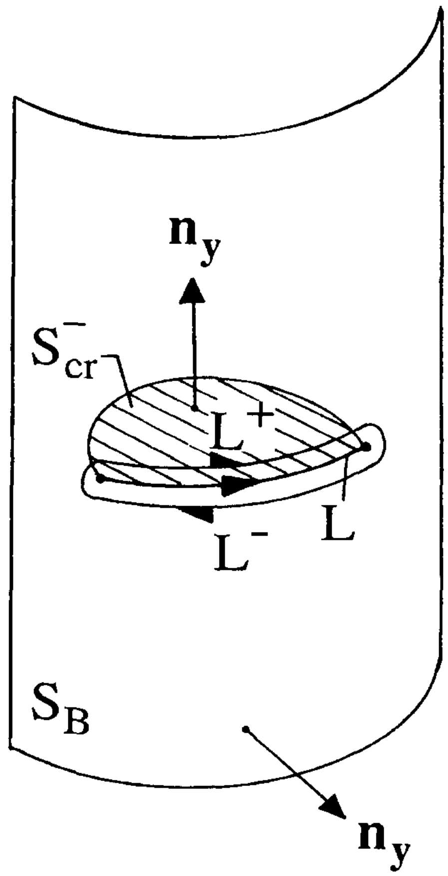

Figure 1. Definition of the closed contour $L^- \cup L^+$ in the case of a surface crack.

in which, by noting that $D_{mnk}(x, y \in L^-) = D_{mnk}(x, y \in L^+)$ and that the orientation of L^- and L^+ are opposite

$$\int_{L^- \cup L^+} \varepsilon_{nlr} u_m(y)\, D_{mnk}(x,y)\, dy_r$$
$$= \int_{L^+} \varepsilon_{nlr}\, \Delta u_m(y)\, D_{mnk}(x,y)\, dy_r$$

From equations (48) and (49), it turns out that, although line integrals appear in the case of a surface crack, their very sum is zero. The theorem is proved.

Notes

(a) In the case of a surface crack, it has been assumed that $u \in C^1(S_B \setminus L)$, not $u \in C^1(S_B)$. Indeed, the surface line L corresponds to an incision in the boundary S_B, giving rise to a *displacement discontinuity* on S_B along L and making the hypothesis $u \in C^1(S_B)$ impossible.
(b) The theorem cannot be proved by considering $S_B \cup S_{cr}$ as a single closed surface. The crack geometry clearly indicates that, contrary to S_{cr}^-, $S_{cr} = S_{cr}^+ \cup S_{cr}^-$ cannot be assumed to belong to the class $C^{1,\alpha}$.

5. Boundary integro-differential equation

It is well-known in isotropy that the BIE obtained in usual way by taking the limit $\Omega \setminus S_{cr} \ni x \to y_0 \in S_{cr}$ in equation (42) is nonunique and insufficient for the determination of the unknowns on the crack. Indeed, the loading on the crack faces is involved in the BIE only through the sum Σt so that, if the stress vector is continuous, i.e., $\Sigma \mathbf{t} = 0$, there is no information about the loading at all. The purpose of this section is to overcome this deficiency by obtaining an adequate BIDE, possibly combined with a BIE.

5.1 Infinite body

Theorem

Under the following assumptions:

(i) the regularity conditions are fulfilled
(ii) $S_{cr}^- \in C^{1,\alpha}$, $0 < \alpha \le 1$
(iii) $\Delta u \in C^{1,\beta}(S_{cr}^-)$, $0 < \beta \le 1$
(from hypotheses (ii) and (iii), it follows that $\Sigma \mathbf{t} = -[C\colon\ (\text{grad}\, \Delta u)] \cdot \mathbf{n}_y^- \in C^{0,\beta'}(S_{cr}^-)$, $0 < \beta' \le 1$) the BIDE writes

$$\forall y_0 \in S_{cr}$$
$$\tfrac{1}{2}\left[t(y_0^-, \mathbf{n}_{y0}^-) - t(y_0^+, \mathbf{n}_{y0}^+)\right]$$
$$= pv \int_{S_{cr}^-} \{D(y, y_0) \Sigma \mathbf{t}(y) + \mathscr{R}(\partial y, y_0, y, \mathbf{n}_y)\, \Delta u(y)\}\, d_y S \cdot \mathbf{n}_{y0}^-$$
$$+ * \int_\Omega D(y, y_0) f(y)\, d_y V \cdot \mathbf{n}_{y0}^- \qquad (50)$$

Proof:

Equations (30) and (31) are valid in view of hypotheses (ii) and (iii), yielding

$$\lim_{x \to y_0^\pm} \int_{S_{cr}^-} D(y,x) \Sigma \mathbf{t}(y)\, d_y S \cdot \mathbf{n}_{y0}^-$$
$$= \mp \tfrac{1}{2} \Sigma \mathbf{t}(y_0) + pv \int_{S_{cr}^-} D(y, y_0) \Sigma \mathbf{t}(y)\, d_y S \cdot \mathbf{n}_{y0}^-$$

and

$$\lim_{x \to y_0^\pm} \int_{S_{cr}^-} \mathscr{R}(\partial y, x, y, \mathbf{n}_y)\, \Delta u(y)\, d_y S \cdot \mathbf{n}_{y0}^-$$
$$= pv \int_{S_{cr}^-} \mathscr{R}(\partial y, y_0, y, \mathbf{n}_y)\, \Delta u(y)\, d_y S \cdot \mathbf{n}_{y0}^-$$

where by $x \to y_0^\pm$ are meant the upper and lower limits with respect to normal $\mathbf{n}_{y0}^-$, respectively. According to the hypotheses, the integral representation (44) of the stress holds. Then multiplying it by $\mathbf{n}_{y0}^-$ and taking the limit as $x \to y_0^\pm$ gives (50). The theorem is proved.

If the loads t^+ and t^- are specified, the solution of the BIDE (50) gives the displacement jump Δu, and then u^+ and u^- using the integral representation of the displacement (42). Conversely, if Δu is prescribed, the BIDE (50) does not allow to obtain t^+ and t^- separately, unless an additional information is supplied, e.g., $t^+ = -t^-$. This is better accounted for by the particular case of the plane crack considered below.

Plane crack in the isotropic medium

From equation (50) immediately results the following corollary.

Corollary

Consider a plane crack imbedded in the infinite medium and lying in the plane $y_3 = 0$. Assuming that:

(i) the medium is isotropic
(ii) the regularity conditions are fulfilled
(iii) $\Delta u \in C^1(S_{cr}^-)$
we have the BIDE for a plane crack:

$$\forall y_0 \in S_{cr}$$
$$\forall \alpha \in \{1,2\},$$
$$\frac{1}{2}[t_\alpha(y_0^-, e_3) - t_\alpha(y_0^+, -e_3)]$$
$$= \frac{1-2\nu}{8\pi(1-\nu)} pv \int_{S_{cr}^-} \frac{1}{r^2} \Sigma t_3(y) r_{,\alpha}\, d_y S$$
$$+ \frac{\mu}{8\pi(1-\nu)}$$
$$\times pv \int_{S_{cr}^-} \frac{1}{r^2} \{(1-2\nu)[r_{,\beta}\, \Delta u_{\alpha,\beta} - r_{,\alpha}\, \Delta u_{\beta,\beta}]$$
$$+ 3 r_{,\alpha} r_{,\beta} r_{,\gamma}\, \Delta u_{\gamma,\beta}\}\, d_y S + W_\alpha(y_0) \qquad (51a)$$

$$
\begin{aligned}
&\frac{1}{2}[t_3(y_0^-, e_3) - t_3(y_0^+, -e_3)] \\
&\quad = -\frac{1-2\nu}{8\pi(1-\nu)} pv \int_{S_{cr}^-} \frac{1}{r^2} \Sigma t_\alpha(y) r_{,\alpha}\, d_y S \\
&\qquad + \frac{\mu}{4\pi(1-\nu)} pv \int_{S_{cr}^-} \frac{1}{r^2} r_{,\alpha}\, \Delta u_{3,\alpha}\, d_y S \\
&\qquad + W_3(y_0)
\end{aligned} \tag{51b}
$$

where

$$
\begin{aligned}
&r = \|y - y_0\|, \\
&r_{,i} = (y_i - y_{0i})/r, \\
&W(y_0) = * \int_\Omega D(y, y_0) f(y)\, d_y V \cdot e_3
\end{aligned}
$$

There is no coupling between $(t_\alpha, \Delta u_\alpha)_{\alpha \in \{1,2\}}$ and $(t_3, \Delta u_3)$ if and only if the loading on the crack is symmetrical, i.e., $\Sigma t = 0$.

The case of the plane crack allows us to go deeper into the question of the aforesaid indetermination of t^+ and t^-. For simplicity, let us consider a circular crack free of body forces ($W = 0$). The crack is symmetrical with respect to its plane $y_3 = 0$. The prescription of Δu_3 can be realized by inserting a rigid wedge, axisymmetrical and *dissymetrical* with respect to the plane $y_3 = 0$, between the crack faces. It is assumed that there is no friction between the wedge and the crack faces, then $\Sigma t_\alpha = 0$, $\alpha \in \{1,2\}$ and equation (51b) can be written in the abbreviated form $(1/2)(t_3^+ - t_3^-) =$ function of Δu_3.

The knowledge of Δu_3 implies that the difference $t_3^+ - t_3^-$ is known, but not t_3^+ and t_3^- separately. This is confirmed when the wedge is turned upside down: the same quantity Δu_3 is prescribed, the difference $t_3^+ - t_3^-$ remains the same, as can be easily verified, but t_3^+ and t_3^- are modified individually. This simple example shows that knowing Δu in equation (50) is not enough to determine the loads t^+ and t^-.

5.2 *Finite body*

Theorem

Consider a finite body Ω containing an embedded or surface crack S_{cr}. Assuming that:

(i) $S_{cr}^- \in C^{1,\alpha}$, $S_B \in C^{1,\alpha'}$, $0 < \alpha, \alpha' \leq 1$
(ii) $\Delta u \in C^{1,\beta}(S_{cr}^-)$, $u \in C^1(S_B)$, $0 < \beta \leq 1$
we have the following system of BIE and BIDE:

$$
\begin{aligned}
&\forall y_0 \in S_B, \\
&\int_{S_{cr}^-} \left[E(y_0, y) \Sigma t(y) + T^T(y_0, y, \mathbf{n}_y)\, \Delta u(y) \right] d_y S \\
&\quad + * \int_{S_B} \left\{ E(y_0, y) t(y, \mathbf{n}_y) \right. \\
&\quad \left. - T^T(y_0, y, \mathbf{n}_y)[u(y) - u(y_0)] \right\} d_y S \\
&\quad + * \int_\Omega E(y_0, y) f(y)\, d_y V = 0
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
&\forall y_0 \in S_{cr} \\
&\tfrac{1}{2}\left[t(y_0^-, \mathbf{n}_{y0}^-) - t(y_0^+, \mathbf{n}_{y0}^+) \right] \\
&\quad = pv \int_{S_{cr}^-} \{ D(y, y_0) \Sigma t(y) \\
&\quad + \mathscr{R}(\partial y, y_0, y, \mathbf{n}_y)\, \Delta u(y) \}\, d_y S \cdot \mathbf{n}_{y0} \\
&\quad + \int_{S_B} \{ D(y, y_0) t(y, \mathbf{n}_y) \\
&\quad - \mathscr{R}(\partial y, y_0, y, \mathbf{n}_y) u(y) \}\, d_y S \cdot \mathbf{n}_{y0}^- \\
&\quad + * \int_\Omega D(y, y_0) f(y)\, d_y V \cdot \mathbf{n}_{y0}^-
\end{aligned} \tag{53}
$$

where in the case of a surface crack, S_B must be replaced by $S_B \setminus L$ throughout.

Proof:

Equation (53) is obtained in a similar way as equation (50). To prove equation (52), let us transform the integral representation of the displacement (43) by means of the rigid body identity (10):

$$
\begin{aligned}
&\forall x \in \Omega \setminus (S_B \cup S_{cr}) \\
&\int_{S_{cr}^-} \left[E(x, y) \Sigma t(y) + T^T(x, y, \mathbf{n}_y)\, \Delta u(y) \right] d_y S \\
&\quad + \int_{S_B} \left\{ E(x, y) t(y, \mathbf{n}_y) \right. \\
&\quad \left. - T^T(x, y, \mathbf{n}_y)[u(y) - u(x)] \right\} d_y S \\
&\quad + * \int_\Omega E(x, y) f(y)\, d_y V = 0
\end{aligned} \tag{54}
$$

Now, we proceed to the limit in equation (54) as $\Omega \setminus (S_B \cup S_{cr}) \ni x \to y_0 \in S_B$. From hypothesis (ii) and relation (18), the integral over S_B is weakly singular while that over S_{cr}^- is regular. Thus, the limit is performed by replacing x by y_0 in equation (54). The theorem is proved.

The solution of the system (52) and (53) give (u, t) on S_B and $(\Delta u, \Sigma t)$ on S_{cr}. The coupling between these equations accounts for the interaction between the outer boundary and the crack. In the case of a surface crack, the system provides no equations along the surface line, since $y_0 \notin L$. However, the lacking equations are compensated for by expressing the displacement compatibility at the surface line. With notations introduce in equation (49), we can write at every point on L:

$$
\Delta u(y_0 \in L) + u(y_0^- \in L^-) - u(y_0^+ \in L^+) = 0
$$

Also, it should be noted that the principal value integral in the BIDE (50) or (53) can be numerically transformed in the manner indicated by Balas et al. (Ref. 19, p. 166) into a regular one which in turn can be easily computed.

6. Conclusions

The specific feature of the formulation has been the general anisotropy of the medium. The results have consisted of:

(1) the limit theorems (25), (30), and (31), which describe the limit behavior of the fundamental solution when the load point x approaches a point y_0 belonging to a given surface S, closed or open. These theorems are generalization of the well-known ones in the isotropic case. It is noteworthy that they have been obtained without even knowing the closed form expression of the fundamental solution. A minimum amount of basic properties, relations (11) to (20), has been enough to complete the proofs.

(2) the BIDEs for crack problems, equation (50) for the infinite body with an embedded crack, and the system of coupled equations, (52) and (53), for a finite body with an embedded or surface crack, which clearly shows the interaction between the outer boundary and the crack.

From the numerical points of view, the success of the boundary formulation in anisotropic problems depends on an efficient computation of the integral representations (11) and (12) of the fundamental solution. Wilson and Cruse[17] have shown that the anisotropic solution can be numerically evaluated with essentially arbitrary accuracy. Further improvements in the numerical scheme should allow to more precisely calculate the fundamental solution as well as to significantly save the CPU time.

Nomenclature

C	fourth-order elasticity tensor
$\mathbf{e}_m$	a unit vector
$\mathbf{t}$	stress vector
$\mathbf{n}_y$	normal vector in the y direction
E	the fundamental displacement tensor
D	the third-order tensor of the fundamental stress
Ω	bounded region
S	boundary
Π	tangent plane passing through y_0
$\mathbf{u}$	displacement vector
I	identity tensor
S_B	outer boundary
S_cr	crack surface

Appendix

Regularization theorem

Let S be a surface (open or closed) with edge ∂S. If:

(i) $S \in C^{1,\alpha}, 0 < \alpha \leq 1$

(ii) $u \in C^1(S)$

then $\forall x \notin S$, $\forall k, l$,

$$\int_S \frac{\partial T_{mk}}{\partial x_l}(x, y, \mathbf{n}_y) u_m(y)\, d_y S = \int_S D_{mnk}(x, y)\mathscr{D}_{nl}(\partial y, \mathbf{n}_y) u_m(y)\, d_y S - \int_{\partial S} \varepsilon_{nlr} D_{mnk}(x, y) u_m(y)\, dy_r$$

where $\mathscr{D}_{nl}$ is the tangent differential operator defined by equation (33). If S is a closed surface, the line integral along ∂S is zero.

Proof:

Relations (7) and (17) give

$$\int_S \frac{\partial T_{mk}}{\partial x_l}(x, y, \mathbf{n}_y) u_m(y)\, d_y S = -\int_S \frac{\partial D_{mnk}}{\partial y_l}(x, y) n_n(y) u_m(y)\, d_y S$$

Now, it is easy to verify the following formula of compound derivatives: for any differentiable functions φ and ψ, $\mathscr{D}_{nl}(\varphi \cdot \psi) = \varphi \cdot \mathscr{D}_{nl}\psi + \psi \cdot \mathscr{D}_{nl}\varphi$. This enables us to transform the kernel as:

$$\begin{aligned} D_{mnk,l}(x, y) n_n(y) u_m(y) &= \mathscr{D}_{nl}(D_{mnk} u_m) - D_{mnk}\mathscr{D}_{nl} u_m \\ &\quad + D_{mnk,n} n_l u_m \quad (i = \partial/\partial y_i) \end{aligned}$$

Hence, invoking the Stokes theorem (Ref. 20, p. 282), which is valid owing to hypotheses (i) and (ii):

$$\forall k, l, \quad \int_S \mathscr{D}_{nl}(D_{mnk} u_m)\, d_y S = \int_{\partial S} \varepsilon_{nlr} D_{mnk} u_m\, dy_r$$

and using the equilibrium equation (9) with $x \neq y \in S$, the theorem is proved.

References

1. Rizzo, F. R. An integral equation approach to boundary value problems of classical elastostatics, *Q. Appl. Math.* 1967, **XXV(1)**, 83–95
2. Cruse, T. A. and Rizzo, F. J. A direct formulation and numerical solution of the general transient elastodynamic problems, *J. Math. Anal. Appl.* 1968, **I**, 244–259
3. Cruse, T. A. A direct formulation and numerical solution of the general transient elastodynamic problems. *J. Math. Anal. Appl.* 1968, **II**, 341–355
4. Cruse, T. A. Numerical solutions in three-dimensional elastostatics. *Int. J. Sol. Struct.* 1969, **5**, 1259–1274
5. Rizzo, F. I. and Shippy, D. J. An advanced boundary integral equation for three-dimensional thermoelasticity. *Int. J. Num. Meth. Eng.* 1977, **11**, 1753–1768
6. Bui, H. D. Applications des potentiels élastiques à l'étude des fissures planes de forme arbitraire en milieu tridimensionel. *Comptes Rendus des Académies des Sciences* (Paris) 1975, **280(A)**, 1157–1160
7. Weaver, J. Three-dimensional crack analysis. *Int. J. Sol. Struct.* 1977, **13**, 321–320

8. Sladek, J and Sladek, V. Three-dimensional curved crack in an elastic body. *Int. J. Sol. Struct.* 1983, **19**, 425–436
9. Sladek, J. and Sladek, V. Dynamic stress intensity factors studied by boundary integro-differential equations. *Int. J. Num. Meth. Eng.* 1986, **23**, 919–928
10. Le van, A. and Royer, J. Integral equations for three-dimensional problems. *Int. J. Fract.* 1986, **31**, 125–142
11. Bui, H. D., Loret, B., and Bonnet, M. Régularisation des équations intégrales de l'élastostatique et de l'élastodynamique. *Comptes Rendus des Académies des Sciences* (Paris) 1985, **300(II, 14)**, 633–636
12. Bui, H. D. and Bonnet, M. Regular BIE for three-dimensional cracks in elastodynamics. *Proceedings of the IUTAM Symposium*, San Antonio, TX, April 1987
13. Polch, E. Z., Cruse, T. A., Huang, C.-J. Buried crack analysis with an advanced traction BIE algorithm. *Proceedings of the Winter Annual Meeting for the ASME*, Miami Beach, FL, November 1985
14. Le van, A. and Royer, J. Theoretical basis of regularized integral equations for elastostatic crack problems. *Int. J. Fract.* 1990, **44**, 155–166
15. Tanaka, M., Sladek, V., and Sladek, J. Regularization techniques applied to boundary integral equation methods. *Appl. Mech. Rev.* 1994, **47(10)**, 457–499
16. Vogel, S. M. and Rizzo, F. J. An integral formulation of three-dimensional anisotropic elastostatic boundary value problems. *J. Elasticity*, 1973, **3(3)**, 203–216
17. Wilson, R. B. and Cruse, T. A. Efficient implementation of anisotropic three-dimensional boundary integral equation stress analysis. *Int. J. Num. Meth. Eng.* 1978, **12**, 1383–1397
18. Sladek, J. and Sladek, V. Three-dimensional crack analysis for an anisotropic body. *Appl. Math. Modelling* 1982, **6**, 374–380
19. Balas, J., Sladek, J., and Sladek, V. *Stress Analysis by Boundary Equation Methods*, Studied in Applied Mechanics 23, Elsevier, New York, 1989
20. Kupradze, V. D., Gegelia, T. G., Basheleishvili, M. O., and Burchuladze, T. V. *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, Series in Applied Mathematics and Mechanics, North Holland, Amsterdam, 1979
21. Willis, J. R. The elastic interaction energy of dislocation loops in anisotropic media. *Q. J. Mech. Appl. Math.* 1965, **17**, 157–174